# Final exam for Econ 671, fall 2013

- This exam is due at 3pm on December 20th; please put it in my (Gray's) mailbox or drop it off at my office.
- You may use textbooks and your notes to answer these questions, but you may not work with other students. You may not look at exam or homework answers from past years either.
- Please write clearly on a separate sheet of paper.
- There are three questions on this exam. Answer all three to the best of your ability. Good luck!

**Problem 1.** Let $X$ and $Y$ be two random variables with the joint density function

$$f(x,y) = \begin{cases} \min(6y, 6-6y, 6x, 6-6x) & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(this looks like a pyramid centered at (1/2, 1/2)).

1. Derive the joint distribution of $X$ and $Y$.
2. Derive the marginal distribution and density of $X$
3. Derive the conditional distribution and density of $Y$ given $X$.
4. Please write an R function to generate draws from this density function. It should take the number of observations to generate as its only argument. Also generate a contour plot of 10000 draws from the function.

**Problem 2.** The Gauss-Markov Theorem states the following:

> Assume that $Y = X\beta + \varepsilon$ where $\varepsilon \sim (0, \sigma^2 I)$ given $X$ ($Y$ and $\varepsilon$ are $n \times 1$ vectors and $X$ is an $n \times k$ matrix with full rank $k$). Then $\hat{\beta} = (X'X)^{-1}X'Y$ is the unique estimator with minimum variance (given $X$) among all linear, unbiased estimators.

(i.e. OLS is BLUE.) Now, suppose the true DGP is

$$Y = X\beta + Z'\alpha + \varepsilon \tag{2}$$

where

- $Y_i$, $X_i$, $Z_i$, and $\varepsilon_i$ are all i.i.d. random variables, $i = 1, \dots, n$,
- $Y_i$, $X_i$, and $Z_i$ are all observed but $\varepsilon_i$ is not,
- $\mathrm{E}(\varepsilon_i \mid X_i, Z_i) = 0$ almost surely,
- $\mathrm{var}(\varepsilon_i \mid X_i, Z_i) = \sigma^2$,
- $\mathrm{E}(Z_i \mid X_i) = 0$ almost surely.

Under these assumptions, the estimator $\hat{\beta} = (X'X)^{-1}X'Y$ is the BLUE of $\beta$, by the Gauss-Markov theorem. But these assumptions also ensure that

$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\alpha} \end{pmatrix} = \begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix}^{-1} \begin{pmatrix} X'Y \\ Z'Y \end{pmatrix} \tag{3}$$

is the BLUE for $(\beta', \alpha')'$, making $\tilde{\beta}$ the BLUE for $\beta$, again by the Gauss-Markov theorem.

How are both estimators the "best linear unbiased estimator" of the same parameter? **Please do not claim that $\hat{\beta}$ and $\tilde{\beta}$ have the same value. They don't.**

**Problem 3.** Suppose that we're interested in the effect of a binary treatment on some outcome of interest. Let $x_i$ denote the (univariate) treatment variable for individual $i$ and $y_i$ the outcome. Suppose that the DGP is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \tag{4}$$

but $\varepsilon_i$ and $x_i$ are potentially correlated. For a concrete example, imagine that $x_i$ indicates whether individual $i$ smokes and $y_i$ is a measure of the individual's overall health.

Fortunately, we have run an experiment that assigned individuals to either be in a treatment or a control group: let $x_i^*$ be 1 when the $i$th individual is in the treatment group and 0 otherwise and assume that

$$\Pr[x_i^* = 0] = \Pr[x_i^* = 1] = 1/2 \tag{5}$$

and $x_i^*$ is set independently of $\varepsilon_i$.

**The complication:** Individuals do not always comply with treatment. In particular,

$$\Pr[x_i = 1 \mid x_i^* = 1] = \Pr[x_i = 0 \mid x_i^* = 0] = 0.90 \tag{6}$$

and (just to complete the algebra for you)

$$\Pr[x_i = 1 \mid x_i^* = 0] = \Pr[x_i = 0 \mid x_i^* = 1] = 0.10. \tag{7}$$

(Maintain this assumption in answering the questions below. You probably wouldn't know this information in real research, but there are times that you would.)

1. Suppose that $x_i^*$ is observed but $x_i$ is not. This is the common situation where the experimenter knows whether or not an individual was treated but not whether or not the individual complied with treatment. Derive the probability limit and asymptotic distribution for the OLS regression of $y_i$ on $x_i^*$ and show that it is inconsistent for $\beta_1$.

2. Based on your answer to the first part, how could you construct (asymptotic) 95% confidence intervals for $\beta_1$? **You do not necessarily need a consistent estimator of $\beta_1$ to construct these intervals.** We know (by fiat) that $x_i^*$ and $x_i$ will different in 10% of the sample on average, so you can work out the worst-case misclassification to derive your answer.

   If you need to impose additional assumptions to derive an answer, feel free to. (It's much better than no answer at all.)

3. Now suppose that you observe $x_i$ but not $x_i^*$. Repeat your steps for the previous two parts, but using $x_i$. Show that the OLS regression of $y$ on $x_i$ is inconsistent for $\beta_1$ and give a procedure for constructing 95% confidence intervals for $\beta_1$ using only data on $y_i$ and $x_i$.

4. Now suppose that both $x_i$ and $x_i^*$ are observed. Prove that $x_i^*$ can be used as an instrument for $x_i$ and show that the two-stage least squares estimator can be used to consistently estimate $\beta_1$ in (4).